"Iterative Substructuring Methods:
the General Elliptic Case"

by

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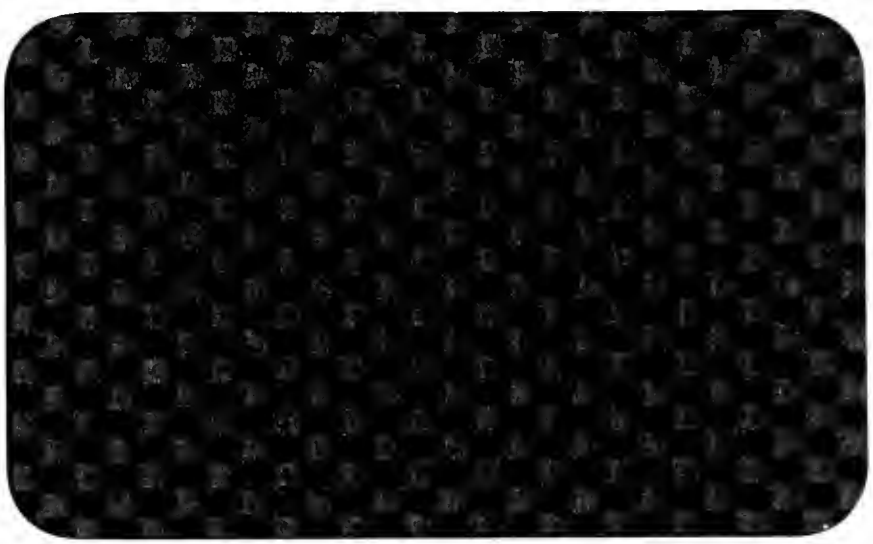

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Abstract

We describe new results on iterative substructuring methods. These are domain decomposition methods for which the subdomains, also known as substructures, do not overlap. The interaction between the subdomains, to satisfy appropriate continuity requirements, is handled by a conjugate gradient method. We first consider the case of two substructures and show that the optimality of a method of this kind can be established by using an extension theorem for finite element spaces. A general extension theorem is then formulated for general conforming finite elements under a mild restriction on the triangulation. The proof of this result extends a technique introduced by Astrakhantsev. It is then established that a mechanism for global transportation of information is necessary in order to obtain fast convergence in the case of many substructures. In the case of a simple second order problem on regions in the plane, an alternative derivation is given of a fast mehod introduced by Bramble, Pasciak and Schatz. Estimates of the rates of convergence of this and an alternative algorithm are also given.

Iterative Substructuring Methods; The General Elliptic Case

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1. Introduction: In this paper, we will consider a family of domain decomposition methods, known as iterative substructuring methods, which are being developed to solve the often very large linear systems of algebraic equations that arise in elliptic finite element work. These methods differ from Schwarz's alternating algorithm in that there is no overlap of the subregions, also called substructures, into which the region is divided. In the basic form of these algorithms, the finite element or finite difference approximations of the elliptic problem restricted to the substructures, are solved repeatedly by a direct method such as Gaussian elimination or, in special cases, a fast Poisson solver. The interaction between the substructures, to satisfy various continuity requirements, is handled by an iterative method typically a preconditioned conjugate gradient method. For a discussion of the use of inexact solvers for the subproblems, see Bramble, Pasciak and Schatz [7] and Dryja, Proskurowski and Widlund, [20,21].

The idea of substructuring of large discrete elliptic problems goes back at least to the early 1960's; see Przemienicki [31]. In industrial practice, as reflected in several large finite element systems such as NORSAM, see Bell, Hatlestad, Hausteen and Araldsen [2], the matrices which represent the interaction between the substructures are computed and they are then factored into triangular factors. In such programs the substructuring idea is also often applied recursively and it has proven a very useful tool for organizing software systems

for which no a priori upper bound on the problem size is contemplated. We note that in such contexts it is important that the modeling, factorization, etc., of substructures can be saved and used again together with one or a few remodeled substructures if the structure is damaged. Similarly the effects of a redesign of part of one of these often very large engineering structures can be modeled at a much more modest expense.

Since the arithmetic work of algorithms of this kind almost invariably grows faster than linearly in the number of degrees of freedom, a divide and conquer strategy, implicit in the entire substructuring philosophy, is quite attractive provided that the interaction phase of the computation can be carried out with a fast iterative method. We note that Petter Bjørstad and Anders Hvidsten of the University of Bergen, have begun an ambitious study of a number of large industrial problems using industrial software modified to allow the use of iterative substructuring methods.

Iterative substructuring methods also show a clear promise for parallel computing. We can expect that finite element codes designed as indicated above will perform quite well without major modifications on parallel systems with a modest number of processors. It can also be plausibly argued that when the number of processors grows, bottlenecks might occur primarily in the part of the computation that involves the interaction between the substructures. The costs of this phase, both in terms of arithmetic and communication, can be considerably decreased by using iterative substructuring methods. No firm conclusions can however be drawn until more experience has been gained in actual

experiments. We note that Keyes and Gropp [24] have already undertaken such a study on an Intel Hypercube.

The case of two or a few substructures, obtained without using intersecting cuts, is by now quite well understood. The first results on optimal methods are now at least five years old; see Bjørstad and Widlund [3,4], Bramble, Pasciak and Schatz [7], Dryja [16], Lebedev [25], Marchuk, Kuznetsov and Matsokin [26] and Widlund [35,36]. A crucial role in this theory is played by an extension theorem for finite element functions which until recently appears to have been known only for Lagrangian elements in the plane and the H^1-norm. In section 2, we formulate a general extension theorem for general conforming finite element spaces in an arbitrary number of dimensions. Our proof, given in Widlund [36], is inspired by the pioneering work of Astrakhantsev [1].

In section 3, we establish, by using elementary arguments, that the performance of iterative substructuring methods must decline in the absence of a mechanism for global transportation of information. In that section, we also consider the simplest case of problems with intersecting cuts, namely second order problems in planar regions approximated by piecewise linear elements. By introducing an idea from multigrid methods, we derive an algorithm which turns out to be identical to one recently introduced by Bramble, Pasciak and Schatz [8]. An alternative algorithm, see Dryja [17] and Dryja, Proskurowski and Widlund [20,21], is also described and estimates are given of the rate of convergence of these two algorithms.

We conclude this introduction with a few additional comments on the literature. Numerical and theoretical work on the case of many

substructures without intersecting cuts are given in Dryja and Proskurowski [18,19]. Bramble, Pasciak and Schatz have designed and analyzed methods for three dimensional problems, see [9]. Together with Richard Ewing they have also developed related methods for local mesh refinements, see Bramble, Ewing, Pasciak and Schatz [6].

There has also been an active interest in algorithms of this kind at Yale. This work has resulted in several papers; see Chan [10], Chan and Reasco [11,12] as well as Keyes and Gropp [24] mentioned earlier. Among the methods considered in these studies is one due to Golub and Mayers [23]. For a discussion of a number of different methods, including one tested very early by Concus, Golub and O'Leary [14] and several suggested by Dihn, Glowinski and Périaux [15]; see section 6 of Bjørstad and Widlund [4].

As shown for example in Marchuk, Kuznetsov and Matsókin [26] and Widlund [36], these methods are closely related to capacitance matrix methods, known in the Soviet literature as fictitious component and fictitious domain algorithms. The resulting systems of equations have much in common with classical potential theory developed around 1900 by Fredholm and others; see Fredholm [22], Proskurowski and Widlund [29,30]. The Soviet literature, with which this author unfortunately is not fully acquainted, also traces the roots of continuous analog of the substructuring mehods back to Poincaré and Steklov, also active around the turn of the century; see Lebedev [25]. It is interesting to note that Lebedev's new book refers to composition rather than decomposition methods. In a case of large engineering structures such as oil platforms or tankers, this is indeed a more natural name in view

of the fact that, in the most concrete sense, these structures must be assembled from substructures.

2. An extension theorem for finite element spaces. We begin this section by considering the structure of the finite element problems and by outlining an iterative substructuring algorithm. We do this in order to provide motivation for the development of an extension theorem.

Let Ω be a Lipschitz region which is the union of two adjacent but nonoverlapping subregions Ω_1 and Ω_2 and Γ_3. Γ_3 is the intersection of the closures of Ω_1 and Ω_2. The boundary of Ω is the union of Γ_D and Γ_N on which essential (Dirichlet) and natural (Neumann) boundary conditions are imposed. The boundary of Ω also equals $\Gamma_1 \cup \Gamma_2$, where the boundary of Ω_1 is $\Gamma_1 \cup \Gamma_3$, etc. A linear, selfadjoint elliptic problem of order 2ℓ is given in variational form,

$$a_\Omega(u,v) = f(v) , \quad \forall\, v \in V_0(\Omega)$$

where $V_0(\Omega)$ is the subspace of the Sobolev space $H^\ell(\Omega)$ with zero Dirichlet data on Γ_D. $f(v)$ is a linear functional obtained from the right hand side of the elliptic equation and the natural boundary conditions by using a Green's formula. We assume that the bilinear form satisfies the following standard conditions,

$$a_\Omega(u,v) = a_\Omega(v,u)$$

$$c||u||^2_{V_0} \leq a_\Omega(u,u)$$

$$|a_\Omega(u,v)| \leq C||u||_{V_0}||v||_{V_0} ,$$

where c and C are positive constants. We can then use the Lax-Milgram theorem to prove the existence of a unique solution. We also note that the discrete elliptic problem, obtained by a Galerkin procedure using a conforming finite element space, will have a positive definite, symmetric stiffness matrix. These properties follow directly from the

conditions above and the fact that the finite element space V^h is a subspace of V.

Associated with the finite dimensional space V^h is a proper triangulation of the subregions Ω_1, Ω_2 (and Ω) with Γ_3 following edges of triangles. The triangulation can be quite general but we insist on a uniform bound on h_K/ρ_K, where h_K is the diameter and ρ_K the radius of the largest inscribed sphere of the element K. No essential complications arise if some of the triangles have curved edges as in isoparametric methods, cf. Ciarlet [13].

Following Ciarlet [13], we define the finite element space in terms of a pair of dual bases $\{\phi_i\}$ and $\{\mu_i\}$ satisfying $\mu_i(\phi_j) = \delta_{ij}$. Following Strang [33], we assume that the basis function ϕ_i is supported on the union of the elements associated with μ_i and that they all are uniform of degree ℓ in the sense that there exist constants C_s, independent of i and the mesh size, such that

$$\max_{x,\,|\alpha|=s} \left|\left(\frac{\partial}{\partial x}\right)^{\alpha} \phi_i(x)\right| \leq C_s h_i^{d_i - s}, \quad s \leq \ell.$$

The exponent d_i is often the degree of a derivative associated with the degree of freedom μ_i. It can be defined naturally by a scaling argument even in other cases, e.g. when μ_i represents an average of the function or one of its derivatives over an element or an edge of an element. The natural interpolation problem for the reconstruction of an element $u_h \in V^h$ involves the quantities $h_i^{d_i}\mu_i(u_h)$.

The linear system of equations corresponding to the discrete elliptic problem has the form,

$$Kx = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} . \tag{2.1}$$

The matrix K_{11} represents the couplings between pairs of degrees of freedom associated with Ω_1 as well as with the intersection of Γ_N and $\bar{\Omega}_1$, K_{13} the couplings between pairs belonging to this set and those associated with Γ_3 etc. The zero blocks in the (1.2) and (2.1) positions reflect the fact that there is no coupling between Ω_1 and Ω_2 when the canonical basis functions are used. The elements of the stiffness matrix K are simply obtained as $k_{ij} = a_\Omega(\phi_i,\phi_j)$ and the right hand sides from $f(\phi_j)$ and the Dirichlet data.

When studying iterative substructuring algorithms, we always assume that it is acceptable to solve discrete problem on the subregions, with some appropriate boundary conditions added on Γ_3. We can therefore concentrate on the case when b_1 and b_2 are zero, since we can reduce the system (2.1) to such a form in a preliminay step. By block Gaussian elimination, this problem is reduced to

$$Sx_3 = (K_{33} - K_{13}^T K_{11}^{-1} K_{13} - K_{23}^T K_{22}^{-1} K_{23}) x_3 = b_3 \tag{2.2}$$

The so called Schur complement is given by $S = S^{(1)} + S^{(2)}$, where $S^{(1)} = K_{33}^{(1)} - K_{13}^T K_{11}^{-1} K_{13}$ is the Schur complement associated with a problem on the subregion Ω_1 ,

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_3 \end{pmatrix} \tag{2.3}$$

Equation (2.3) is the finite element approximation of the elliptic

problem restricted to Ω_1, with natural boundary conditions added on Γ_3. The elements of $K_{33}^{(1)}$ are of the form $a_\Omega(\phi_i,\phi_j)$, where ϕ_i and ϕ_j are basis functions associated with degrees of freedom on Γ_3. We note that K_{33} in (2.1) naturally can be written as $K_{33}^{(1)} + K_{33}^{(2)}$, the sum of matrices which represent the contributions to the integral $a_\Omega(\phi_i,\phi_j)$ from Ω_1 and Ω_2, respectively.

It has been shown, see e.g. Bjørstad and Widlund [3,4] or Widlund [35], that the condition number of S goes to infinity when the problem size increases. A preconditioner should therefore be found. It is not surprising that $S^{(1)}$ is a good preconditioner for S in a conjugate gradient algorithm to solve (2.2). The rate of convergence is determined by the stationary values of the generalized Rayleigh quotient

$$x_3^T S x_3 \;/\; x_3^T S^{(1)} x_3 \;.$$

In particular, it is important to establish that these eigenvalues lie in a fixed interval. The Rayleigh quotient is bounded from below by one since $S = S^{(1)} + S^{(2)}$ and $S^{(2)}$ is positive definite, just as $S^{(1)}$. An upper bound of (1+C) can be established by showing that

$$x_3^T S^{(2)} x_3 \leq C x_3^T S^{(1)} x_3 \;, \quad \forall\, x_3 \tag{2.4}$$

Since b_1 and b_2 are assumed to be zero in equation (2.1), it is natural to call the corresponding finite element function a piecewise, discrete harmonic function. It is easy to show that the strain energy x^TKx of such a function is equal to $x_3^T S x_3$. Similarly $x_3^T S^{(1)} x_3$ and $x_3^T S^{(2)} x_3$ are the strain energies of discrete harmonic functions defined on Ω_1 and Ω_2 respectively, with common boundary values on Γ_3, represented by x_3.

Of all elements in $H^1(\Omega)$ which take on some specific values on $\partial\Omega$, the solution of Laplace's equation provides the minimal strain energy. This can easily be shown by using a Green's formula. The same result holds for any conforming finite elements. Therefore, to establish the bound (2.4), we can use an analog of a well known extension theorem for Sobolev spaces; cf. e.g. Stein [32]. We assume that the complement of the region Ω also has been triangulated in an equally benign way and that $V^h(\Omega)$ has been extended to $V^h(R^n) \subset H^\ell(R^n)$.

Theorem 1. Let Ω be a Lipschitz region and let $V^h(\Omega)$ be a conforming finite element space spanned by basis functions satisfying the uniformity condition given above. Then there exists a constant $C(\Omega)$, which depends on the Lipschitz constant of the boundary of Ω, ℓ, n, the choice of the particular finite element space and $\max(h_k/\rho_k)$ but not on the dimension of V^h and a linear operator E^h which maps $V^h(\Omega)$ into $V^h(R^n)$ such that,

$$E^h u_h\big|_\Omega = u_h \ , \qquad \forall\ u_h \in V^h(\Omega)$$

and

$$||E^h u_h||_{H^\ell(R^n)} \leq C(\Omega)\ ||u_h||_{H^\ell(\Omega)} \ .$$

A proof is given in Widlund [36]. We note that its key ingredients are the extension theorem for Sobolev spaces and the tools developed by Strang [33] to approximate general elements of $H^\ell(R^n)$ by finite element functions.

The proofs given previously in Bjørstad and Widlund [4] and Bramble, Pasciak and Schatz [7] were limited to Lagrangian and linear finite elements respectively to plane regions and to the case $\ell = 1$.

These proofs, virtually identical, use Bramble-Hilbert's lemma, Sobolev's lemma and a regularity result for elliptic problems on regions with corners.

We note that it is easy to adopt Theorem 1 to obtain a bound for the iterative substructuring algorithm considered above. We need to extend a finite element function u_h defined on Ω_1 and vanishing on $\Gamma_D \cap \Gamma_1$ into a function on Ω which vanishes on Γ_D. First extend u_h by zero into bounded subregions of the complement of Ω_1 which are bordered by the components of Γ_D. This step leaves the strain energy unchanged and we can now use the theorem given above to extend the finite element function to all of R^n. Clearly the restriction of this function to Ω_2 provides the extension necesary to establish inequality (2.4) and the optimality of the conjugate gradient algorithm.

3. Many Substructures. The study of iterative substructuring methods for problems with many substructures is quite important since we wish to find out if these algorithms are of real promise for parallel computers with many processors. We are of course primarily interested in algorithms with a performance which does not deteriorate very much when the region is partitioned into an increasing number of subregions. The algorithm described in the previous section works very well for two or a few substructures and it can also be used for regions partitioned into any number of strips. However in each iteration, information is exchanged only between neighboring substructures across the interfaces. As demonstrated in numerical experiments by Dryja and Proskurowski [18], [19], the rate of convergence will suffer when the region is divided into an increasing number of strips. It is in fact easy to see that a residual of modestly low frequency at one end of a region will affect the error everywhere and that therefore the number of steps until convergence of any iterative method must in general exceed the number of strips if the interaction is solely through next neighbors. A similar argument is valid for general regions. Speedup can only be obtained by introducing a mechanism for global transportation of information across the region.

A solution is provided by considering an idea from multigrid theory; see Yserentant [37]. Let us for simplicity consider only piecewise linear elements and Poisson's equation, in which case

$$a_\Omega(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx = (u,v)_{H^1(\Omega)} ,$$

and a region in the plane divided into triangular substructurs Ω_i ,

each of which in turn is divided into triangular elements. We only consider the Dirichlet case and use the standard notation for the norms and seminorms of Sobolev spaces. As in section 2, we avoid very thin triangles in both of these triangulations. The typical diameters associated with the two partitions are H and h respectively.

As explained in section 2, we can work throughout with piecewise, discrete harmonic functions. Any such function u_h can be partitioned into

$$u_h = I_H u_h + (u_h - I_H u_h) \tag{3.1}$$

where I_H is the interpolation operator associated with the coarse mesh, cf. Yserentant [37]. It is easy to show that the piecewise linear function $I_H u_h$ as well as $u_h - I_H u_h$ are piecewise, discrete harmonic. Since the second term in (3.1) vanishes at all vertices of the coarse mesh, it represents higher frequencies; the pitch of a drum increases substantially if held fixed at a number of points. A preliminary preconditioner is introduced in terms of the bilinear form,

$$(I_H u_h, I_H v_h)_{H^1(\Omega)} + (u_h - I_H u_h, v_h - I_H v_h)_{H^1(\Omega)} \ . \tag{3.2}$$

The resulting linear system of equations is block diagonal if we work with the standard basis functions on the coarse mesh and the standard ones for the fine mesh associated with all the vertices except those which are also a vertex of a large triangle. One of the blocks thus corresponds exactly to the original problem solved on the coarse mesh. If this system of equations is solved directly, then it provides the

necessary, quite modest amount of global information. The second term should be modified since by itself it gives rise to a linear system which is almost as difficult as the original one.

Before this is done it is important to note that for $v_h = u_h$ the expression (3.2) is bounded from below by $\frac{1}{2} |u_h|^2_{H^1(\Omega)}$. This follows from the triangle inequality. The first term $|I_H u_h|^2_{H^1(\Omega)}$ is bounded by $C||u_h||_{L^\infty(\Omega)}$ and we finally obtain

$$|I_H u_h|^2_{H^1(\Omega)} \leq C(1 + \log(H/h)) \, |u_h|^2_{H^1(\Omega)}. \qquad (3.3)$$

The same type of bound can also easily be established for $|u_h - I_H u_h|^2_{H^1(\Omega)}$ by using the triangle inequality. The estimate (3.3) follows from an inequality for finite element functions in the plane that does not have a direct continuous counterpart,

$$||u_h||_{L^\infty(\Omega_i)} \leq C\{(1+\log(H/h)) \, |u_h|^2_{H^1(\Omega_i)} + 1/H^2 \, ||u_h||^2_{L^2(\Omega_i)}\} \; ; \; (3.4)$$

see Bramble [5], Bramble, Pasciak and Schatz [8], Rukhovets and Oganesyan [27], Thomée [34] or Yserentant [37]. This result is often given only for piecewise linear functions but it also holds for more general finite elements. Note that $\log(H/h)$ measures the number of refinement steps needed to go from the substructure level to the fine triangulation.

The estimate (3.4) cannot be improved, but the L_2-term can be removed for the case at hand by noting that

$$|I_h(u_h+\text{const.})|_{H^1(\Omega_i)} = |I_H u_h + \text{const.}|_{H^1(\Omega_i)} = |I_H u_h|_{H^1(\Omega_i)}$$

and that therefore Poincaré's inequality can be applied.

We can therefore conclude that the preconditioner given by (3.2) gives a conjugate gradient mehod with a condition number bounded by $C(1 + \log(H/h))$ and that it therefore would converge in $C \log(1/\varepsilon)\ (1+\log(H/h))^{1/2}$ steps. If we destroyed the mechanism for global transportation of information by replacing the coarse finite element model, defined by the first term of (3.2), by a diagonal matrix, then the condition number estimate would increase by a factor $(1/H)^2$ and the number of iteration steps to $C \log(1/\varepsilon)\ (1+\log(H/h))^{1/2}(1/H)$.

As in the previous section, we can associate a strain energy with each individual substructure. The estimates given below will be for individual substructures and the final results will simply be obtained by adding the contributions from all of them. In Ω_i the finite element function $w_h = u_h - I_H u_h$ depends only on its boundary values since it is discrete harmonic in Ω_i. We will obtain our final preconditioner by replacing $|w_h|^2_{H^1(\Omega_i)}$, one of the contributions from Ω_i to the preliminary preconditioner (3.2), by the square of a norm of these boundary values. Before we proceed, we introduce a second optional preconditioner for the case of two substructures. It has been established in Bjørstad and Widlund [4] that $S^{(1)}$, defined in section 2, is spectrally equivalent to J. This operator is defined in terms of its square J^2 which is a finite element discretization of $-(\frac{d}{ds})^2$ on Γ_3 with Dirichlet conditions at the end points. In this construction, we use the mesh and finite element space obtained by restricting the given finite element space to the curve Γ_3. As shown in Bjørstad and Widlund [4], the quadratic form $x_3^T J x_3$ is equivalent to

$$||u_h||^2_{H_{00}^{1/2}(\Gamma_3)} = |u_h|^2_{H^{1/2}(\Gamma_3)} + \int_0^{\ell_3} \left(\frac{|u_h(x(s))|^2}{s} + \frac{|u_h(x(s))|^2}{\ell_3 - s} \right) ds \,, \quad (3.5a)$$

where ℓ_3 is the length of Γ_3. The seminorm $|u_h|_{H^{1/2}(\Gamma_3)}$ is given by

$$|u_h|^2_{H^{1/2}(\Gamma_3)} = \int_0^{\ell_3} \int_0^{\ell_3} \frac{|u(x(s)) - u(x(s'))|^2}{|s-s'|^2} \, ds \, ds' \,. \quad (3.5b)$$

Let $\partial\Omega_i = \bigcup_j \Gamma_{ij}$, where $\Gamma_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$ $i \neq j$, an edge of the triangular substructure Ω_i. Since we know how to work with the operator J and w_h vanishes at the vertices of the substructures, it is natural to replace $|w_h|^2_{H^1(\Omega_i)}$ in the preconditioner by

$$\sum_j ||w_h||^2_{H_{00}^{1/2}(\Gamma_{ij})} \,. \quad (3.6)$$

The expression (3.6) provides an upper bound for $|w_h|^2_{H^1(\Omega_i)}$. This is so since it is known that any element in $H_{00}^{1/2}(\Gamma_{ij})$ can be extended by zero to the rest of $\partial\Omega_i$ with the resulting function in $H^{1/2}(\partial\Omega_i)$. The $H^1(\Omega_i)$-norm of the discrete harmonic function with these boundary values can be bounded by the $H_{00}^{1/2}(\Gamma_{ij})$ norm of its trace on Γ_{ij}. This follows from the extension theorem given in Section 2; for details see Bjørstad and Widlund [4]. Since w_h can be written as the sum of three functions of this kind we obtain,

$$|w_h|^2_{H^1(\Omega_i)} \leq C \sum_j ||w_h||^2_{H_{00}^{1/2}(\Gamma_{ij})} \,. \quad (3.7)$$

To obtain an upper bound for $\sum_j ||w_h||^2_{H_{00}^{1/2}(\Gamma_{ij})}$, we use the definition (3.5). We use the trace theorem and find that

$$\sum_j |w_h|^2_{H^{1/2}(\Gamma_{ij})} \leq |w_h|^2_{H^{1/2}(\partial\Omega_i)} \leq C(|w_h|^2_{H^1(\Omega_i)} + 1/H^2 \, ||w_h||^2_{L^2(\Omega_i)}) \,.$$

Since the seminorm of the left hand side does not change if we add a constant to w_h, we can use Poincaré's inequality and the bound obtained previously for $|w_h|_{H^1(\Omega_i)}$ to obtain,

$$\sum_j |w_h|^2_{H^{1/2}(\Gamma_{ij})} \leq C|w_h|^2_{H^1(\Omega_i)} \leq C(1 + \log(H/h))\ |u_h|^2_{H^1(\Omega_i)} . \quad (3.8)$$

It is also straightforward to estimate the second term of (3.5a). By using that w_h vanishes at the vertices of Ω_i, we find that

$$\begin{aligned} \int_0^{\ell_{ij}} \frac{|w_h(x(s))|^2}{s}\, ds &= \int_0^{h_{ij}} \frac{|w_h(x(s))|^2}{s}\, ds + \int_{h_{ij}}^{\ell_{ij}} \frac{|w_h(x(s))|^2}{s}\, ds \\ &\leq \frac{|w_h(x(h_{ij}))|^2}{h^2} \int_0^{h_{ij}} s\, ds + ||w_h||^2_{L^\infty(\Omega_i)} \log (d_{ij}/h_{ij}) \quad (3.9) \\ &\leq C(1 + \log(H/h))||w_h||^2_{L^\infty(\Omega_j)} \leq C(1 + \log(H/h))\ ||u_h||^2_{L^\infty(\Omega_i)} . \end{aligned}$$

Here h_{ij} is the distance to the first mesh point along Γ_{ij} and ℓ_{ij} the length of Γ_{ij}. The last inequality follows easily since $||I_H u_h||_{L^\infty(\Omega_i)} \leq ||u_h||_{L^\infty(\Omega_i)}$.

We obtain from (3.4), (3.8) and (3.9) that

$$\sum_j ||w_h||^2_{H^{1/2}_{00}(\Gamma_{ij})} \leq C(1+\log(H/h))^2(|u_h|^2_{H^1(\Omega_i)} + 1/H^2\ ||u_h||^2_{L^2(\Omega_i)}) , \quad (3.10)$$

and finally, since $w_h = (I-I_H)u_h = (I-I_H)(u_h+\text{const.})$, the following bounds are established by another application of Poincaré's inequality,

$$\begin{aligned} c|u_h|^2_{H^1(\Omega_i)} &\leq |I_H u_h|^2_{H^1(\Omega_i)} + \sum_j ||u_h - I_H u_h||^2_{H^{1/2}_{00}(\Gamma_{ij})} \\ &\leq C(1 + \log(H/h)^2)\ |u_h|^2_{H^1(\Omega_i)} . \end{aligned}$$

Using standard technical tools, we have just provided is a simplified proof of the main result in Bramble, Pasciak and Schatz [8].

Theorem 2. The preconditioned conjugate gradient method which uses the preconditioner defined by

$$(I_H u_h, I_H v_h)_{H^1(\Omega)} + c_1 \sum_{i,j} ((u_h - I_H u_h,\ v_h - I_H v_h))_{H_{00}^{1/2}(\Gamma_{ij})}$$

converges in $C \log(1/\varepsilon)\cdot(1 + \log(H/h))$ steps. Here c_1 is any constant on the order of one.

The same tools can be used to prove an equally strong bound for a natural extension of the method discussed in section 2. This algorithm is described more fully in Dryja [17] and Dryja, Prokurowski and Widlund [20],[21]. In brief, we divide the substructures into two disjoint sets. We denote the corresponding unions by $\Omega^{(1)}$ and $\Omega^{(2)}$. These sets will play the same role as Ω_1 and Ω_2 in the algorithm described in section 2. We must assume that the substructuring is such that if two substructures Ω_i and Ω_j have a common edge $\Gamma_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$, then Ω_i and Ω_j do not belong to the same subset. The set $\bar{\Omega}^{(1)}$ is thus a union of the closure of certain substructures, and these substructures intersect only at vertices of substructures. We can formulate a Neumann problem on $\Omega^{(1)}$ and it is possible to reduce the corresponding large linear system of equations to linear systems corresponding to individual substructures by using Lagrange multipliers; for details see Dryja [17] and Dryja, Proskurowski and Widlund [20], [21]. Just as described in section 2, the solution of the linear system which corresponds to the preconditioner amounts to solving a Neumann problem on $\Omega^{(1)}$ followed by a Dirichlet problem on

$\Omega^{(2)}$ using the boundary values obtained from the Neumann problem as Dirichlet data. In the case at hand, $\Omega^{(2)}$ is the disjoint union of a number of substructures. These problems are therefore independent and can be solved in parallel. We also note that global transportation of information is assured by the fact that $\overline{\Omega}^{(1)}$ is connected.

As in section 2, the rate of convergence is determined by an upper bound for

$$x_3^T S x_3 / x_3^T S^{(1)} x_3 ,$$

i.e. by bounding the strain energy associated with $\Omega^{(2)}$ in terms of the energy of $\Omega^{(1)}$. This can be accomplished by estimating the strain energy of one substructure belonging to the second set by the strain energy of those neighbors which belong to the first set.

In our proof we again partition u_h as in (3.1). The strain energy attributable to $I_H u_h$ can be handled straightforwardly by using inequality (3.3) for individual subregions. From inequality (3.10) follows that

$$||u_h - I_H u_h||^2_{H_{00}^{1/2}(\Gamma_{ij})} \leq C(1 + \log(H/h)^2) \, |u_h|^2_{H^1(\Omega)} ,$$

i.e. the square of the $H_{00}^{1/2}$-norm of the boundary data of $u_h - I_H u_h$ on one edge of a region can be bounded by the strain energy of one of its next neighbors at the expense of a factor $(1 + \log(H/h)^2)$. The use of estimate (3.7) now completes the proof of the following theorem.

Theorem 3. The preconditioned conjugate gradient method

introduced by Dryja, Proskurowski and Widlund converges in $C \log(1/\varepsilon)(1 + \log(H/h))$ steps.

We note in conclusion that this theory has been extended to more general families of finite element spaces; see Dryja [17] and Dryja, Proskurowski and Widlund [21].

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